

SELBERG INTEGRAL AND MULTIPLE ZETA VALUES

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1. INTRODUCTION

In this paper, we show that the coefficients of the Talor expansion of Selberg integrals with respect to its exponent variables are expressed as a linear combinations of multiple zeta values. First object we treat is Selberg integral, the peirod integrals of abelian coverings of the moduli spaces of n -points in \mathbf{P}^1 . Let $2 \leq r \leq n$ be integers, α_{ij} be positive real numbers. For an element $f \in \mathbf{C}[\frac{1}{x_i - x_j}]$, the function on $D = \{x_1 < x_r < \cdots < x_3 < x_2\}$ defined by the integral

$$\int_{D'} f \prod_{i < j} (x_j - x_i)^{\alpha_{ij}} dx_{r+1} \cdots dx_n,$$

where $D' = \{x_1 < x_n < \cdots < x_r\}$ is called a Selberg integral. It is considered as a family of period integrals for abelian coverings of the moduli space of distance n -points in \mathbf{C} . It is a function on x_1, \dots, x_r and exponent parameters α_{ij} . If $r = 2$, by the simple functional equation on x_1 and x_2 , the Selberg integral is determined by its restriction to $x_1 = 0$ and $x_2 = 1$. This restricted function on the exponent parameter α_{ij} is called $n - 2$ -dimensional Selberg integral of 0-variables. It is natural to think that this period integral is equipped with an arithmetic nature.

The second object in our paper is multiple zeta value introduced by Euler. Let $\mathbf{k} = (k_1, \dots, k_m)$ be a sequence of integers such that $k_i \geq 1$ ($i = 1, \dots, m - 1$) and $k_m \geq 2$. The multiple zeta value for the index \mathbf{k} is define by

$$\zeta(\mathbf{k}) = \sum_{n_1 < \cdots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}.$$

The natural number $|\mathbf{k}| = \sum_{p=1}^m k_p$ is called the weight of the index \mathbf{k} . The integer $|\mathbf{k}|$ is called the weight of the multiple zeta value $\zeta(\mathbf{k})$. By using the iterated integral expression, multiple zeta values are regarded as period integrals for the fundamental group $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$ of $\mathbf{P}^1 - \{0, 1, \infty\}$. (See §2.1 for the iterated integral expression of multiple zeta values.) Notice that the motivic weight of $\zeta(\mathbf{k})$ is equal to $-2|\mathbf{k}|$. The main theorem of this paper is

Theorem 1.1. *For a suitable choice of f , the degree w coefficient of the Talor expansion for α_{ij} of the Selberg integral of 0-variables is a linear combination of weight w multiple zeta values.*

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Let us illustrate a primitive example for this statement. By the well known equality

$$\log \Gamma(1+x) = \gamma x + \sum_{n \geq 2} \frac{\zeta(n)x^n}{n},$$

we have

$$\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} = \exp\left(\sum_{n \geq 2} \frac{\zeta(n)(\alpha^n + \beta^n - (\alpha+\beta)^n)}{n}\right).$$

In this example, we choose $r = 2, n = 3$ and $f = \alpha/(1-x)$. (See [2] [6] for another expression of this quantity.) We can find the prototype of this theorem in [2]. The method of the choice of f leads us to an interesting combinatorial problem. In this paper, we will answer to this problem. This choice of f happens to be equal to β -nbc base after Falk-Terao [3].

Let us summerize the method of the proof of the main theorem. Let $\mathbf{C}\langle\langle X, Y \rangle\rangle$ be the formal non-commutative free algebra generated by X and Y . After Drinfeld, associator $\Phi(X, Y)$ is defined as an element of $\mathbf{C}\langle\langle X, Y \rangle\rangle$. It is known that the coefficient of $\Phi(X, Y)$ is expressed as a \mathbf{Q} -linear combination of multiple zeta values by Le-Murakami [5].

Let $\mathbf{Q}[[\alpha_i]]$ and $\mathbf{C}[[\alpha_i]]$ be formal power series rings with variables α_i ($i \in I$) over \mathbf{Q} and \mathbf{C} respectively. Let $\rho : \mathbf{C}\langle\langle X, Y \rangle\rangle \rightarrow M(r, \mathbf{C}[[\alpha_i]])$ be a continuous homomorphism, where all the matrix elements of $\rho(X)$ and $\rho(Y)$ are degree 1 homogeneous in $\mathbf{Q}[[\alpha_i]]$. By the result of Le-Murakami, the coefficients of the Talor expansion of any matrix element $\rho(\Phi(X, Y))$ are expressed by multiple zeta function. For any solution s of the differential equation

$$ds = \left(\frac{\rho(X)}{x} + \frac{\rho(Y)}{x-1}\right)sdx,$$

we have

$$\lim_{x \rightarrow 1} ((1-x)^{-\rho(Y)} s(x)) = \rho(\Phi(X, Y)) \lim_{x \rightarrow 0} (x^{-\rho(X)} s(x)).$$

In this paper, we construct representation ρ and horizontal section s with the following properties.

1. All the element of $\lim_{x \rightarrow 0} (x^{-\rho(X)} s(x))$ is expressed as $n-3$ -dimensional 0-vairalbe Selberg integrals by taking a limit for some of $\alpha_i \rightarrow 0$.
2. All the element of $\lim_{x \rightarrow 1} ((1-x)^{-\rho(Y)} s(x))$ is expressed as $n-2$ -dimensional 0-vairalbe Selberg integrals by taking the same limit $\alpha_i \rightarrow 0$.

For the construction of representation with these properties, we make a combinatorial preparation in Section 4. The construction of ρ depends on the computation of the higher direct image of the local system on the moduli space X_n of distinct n -points in \mathbf{C} for the projection $X_n \rightarrow X_{n-1}$. This computation is executed in Section 3. The limit for $x \rightarrow 0$, $x \rightarrow 1$ and $\alpha_i \rightarrow 0$ are given in Section 5.

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2. PRELIMINARY

2.1. Drinfeld Associator. In this section, we recall known facts about Drinfeld associator. Let $R = \mathbf{C}\langle\langle X, Y \rangle\rangle$ be the completion of non commutative polynomial ring of symbols X, Y with respect to its total degree on X and Y . Let $V = R$. Then X and Y acts on V as the left multiplication and under this action X and Y are regarded as elements in $End_{\mathbf{C}}(V)$. Now we consider a differential form ω on $\mathbf{C} - \{0, 1\}$ with the coefficient in $End_{\mathbf{C}}(V)$ defined as

$$\omega = \frac{X}{x}dx + \frac{Y}{x-1}dx,$$

where x is the coordinate of \mathbf{C} . Let $E(x) = \exp(\int_{x_0}^x \omega)$ be the solution of the differential equation for $End_{\mathbf{C}}(V)$ -valued function $E(x)$

$$dE(x) = \omega E(x)$$

with the initial condition $E(x_0) = 1$. Then by the standard argument for iterated integrals, $\exp(\int_{x_0}^x \omega)$ is expressed as

$$(2.1) \quad \exp(\int_{x_0}^x \omega) = 1 + \int_{x_0}^x \omega + \int_{x_0}^x \omega \omega + \cdots$$

Here we use the convention for iterated integrals defined by the inductive relation

$$\int_p^q \omega_1 \cdots \omega_n = \int_p^q (\omega_1(q_1)) \int_p^{q_1} \omega_2 \cdots \omega_n.$$

The expression (2.1) implies, $\exp(\int_{x_0}^x \omega) \in \mathbf{C}\langle\langle X, Y \rangle\rangle^\times$ and the shuffle relation for iterated integral implies that $E = \exp(\int_{x_0}^x \omega)$ is a group like element, i.e. $\Delta(E) = E \otimes E$ in $\mathbf{C}\langle\langle X, Y \rangle\rangle \hat{\otimes} \mathbf{C}\langle\langle X, Y \rangle\rangle$ where the comultiplication $\Delta : \mathbf{C}\langle\langle X, Y \rangle\rangle \rightarrow \mathbf{C}\langle\langle X, Y \rangle\rangle \hat{\otimes} \mathbf{C}\langle\langle X, Y \rangle\rangle$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$. The set $\hat{G} = \{\Delta(g) = \Delta(g) \otimes \Delta(g) \mid g \in \mathbf{C}\langle\langle X, Y \rangle\rangle^\times\}$ is called the set of group like elements and closed under the multiplication. By the theory of differential equation only with regular singularity, the limit $\lim_{x \rightarrow 1} \exp(\int_x^0 \frac{Y}{x-1} dx) \exp(\int_{x_0}^x \omega)$ exists. In the same way, the limit

$$\Phi(X, Y) = \lim_{x \rightarrow 1, y \rightarrow 0} \exp(\int_x^0 \frac{Y}{x-1} dx) \exp(\int_y^x \omega) \exp(\int_1^y \frac{X}{x} dx)$$

exists and contained in $\mathbf{C}\langle\langle X, Y \rangle\rangle^\times$. $\Phi(X, Y)$ is called the Drinfeld associator. Since $\exp(\int_x^0 \frac{Y}{x-1} dx)$ and $\exp(\int_1^y \frac{X}{x} dx)$ are elements in \hat{G} and \hat{G} is a closed subset of $\mathbf{C}\langle\langle X, Y \rangle\rangle^\times$, the limit $\Phi(X, Y)$ is an element in \hat{G} .

We recall that the relation between multiple zeta values and the coefficients of the Drinfeld associator. Firstly, we recall the definition of multiple zeta values. Let k_1, \dots, k_n be integers such that $k_i \geq 1$ for $i = 1, \dots, n$ and $k_n \geq 2$. Set $\mathbf{k} = (k_1, \dots, k_n)$. The following series

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_n) = \sum_{m_1 < m_2 < \cdots < m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

is called the multiple zeta values for the index $\mathbf{k} = (k_1, \dots, k_n)$. The number $|\mathbf{k}| = \sum_{i=1}^n k_i$ is called the weight of the index \mathbf{k} . Let L_w be the finite dimensional \mathbf{Q} vector subspace of \mathbf{C} generated by $\zeta(\mathbf{k})$, with the weight $|\mathbf{k}| = w$. The following iterated integral expression of multiple zeta values is fundamental.

$$\begin{aligned} \zeta(k_1, \dots, k_n) = & \int_0^1 \underbrace{\frac{dx}{x} \dots \frac{dx}{x}}_{k_n-1} \frac{dx}{1-x} \underbrace{\frac{dx}{x} \dots \frac{dx}{x}}_{k_{n-1}-1} \frac{dx}{1-x} \\ & \dots \underbrace{\frac{dx}{x} \dots \frac{dx}{x}}_{k_1-1} \frac{dx}{1-x}. \end{aligned}$$

By using this expression and shuffle relation, for elements a and b in L_{w_1} and L_{w_2} , we can show that ab is an element in $L_{w_1+w_2}$. Using this fact, we define the homogeneous multiple zeta value ring (homogeneous MZV ring for short) H in $\mathbf{C}\langle\langle X, Y \rangle\rangle$ by

$$H = \bigoplus_{w \geq 0} \bigoplus_{W: \text{word of length } w \text{ on } X, Y} L_w \cdot W.$$

The following proposition is due to Le-Murakami [5].

Proposition 2.1. $\Phi(X, Y) \in H$.

It is very useful to specialize this universal result to a special class of representations of $\mathbf{C}\langle\langle X, Y \rangle\rangle$. Let R be a homogeneous complete ring generated by degree 1 elements over \mathbf{Q} , i.e. R is generated topologically by degree 1 homogeneous elements $\alpha_1, \dots, \alpha_m$ with homogeneous relations and complete under the topology defined by its degree. The decomposition of R with respect to its degree is denoted by $R = \hat{\bigoplus}_{d \geq 0} R_d$. Let $\mathbf{R}_{\mathbf{C}}$ be the completion of $R \otimes \mathbf{C}$ with respect to the topology defined by its degree. A ring homomorphism $\rho : \mathbf{C}\langle\langle X, Y \rangle\rangle \rightarrow M(r, \mathbf{R}_{\mathbf{C}})$ is called a homogeneous rational representation of degree 1 if and only if all the matrix elements of $\rho(X)$ and $\rho(Y)$ are degree 1 homogeneous elements in R . The homogeneous MZV ring H_R for R is defined by $H_R = \hat{\bigoplus}_{d \geq 0} (R_d \otimes L_d)$. The following corollary is a direct consequence of Proposition 2.1.

Corollary 2.2. *Let $\rho : \mathbf{C}\langle\langle X, Y \rangle\rangle \rightarrow M(r, \mathbf{R}_{\mathbf{C}})$ be a homogeneous rational representation of degree 1. Then all the matrix elements of $\rho(\Phi(X, Y))$ are elements in H_R .*

3. SELBERG INTEGRAL

3.1. Combinatorial aspects. Let $[n] = \{1, \dots, n\}$. A graph Γ consists of the set of vertices V_Γ and edges E_Γ . We assume that every edge has distinct two terminals. Moreover we assume for any two vertices p and q , there exists at most one edge whose terminals are p and q . An edge is written as (p, q) , where p and q are its terminals. For a graph Γ , we can associate a 1-dimensional simplicial complex by usual manner and we use the standard terminology connected component, tree, and so on. Moreover, if the order of

E_Γ is specified, it is called ordered graph. A specified point for each connected components is called a root. The specified point for a connected component is called the root of the connected component and the set of roots is denoted by $R = R_\Gamma$. For two sets V, R such that $R \subset V$, we define $\Omega^i(V \bmod R)$ by $\wedge^i(\Omega_{X_V}^1/p^*\Omega_{X_R}^1)$, where $X_V = \{(x_i)_{i \in V} \mid x_i \neq x_j \text{ for } i \neq j\}$, $X_R = \{(x_i)_{i \in R} \mid x_i \neq x_j \text{ for } i \neq j\}$, and p is the natural projection $X_V \rightarrow X_R$. Then it is easy to see that $\Omega^{\#V-\#R}(V \bmod R)$ is a rank 1 \mathcal{O}_{X_n} module generated by $\wedge_{i \in V-R} dx_i$. For an edge $e = (p, q), p, q \in V_\Gamma$, we define $\omega_e = d \log(x_p - x_q) \in \Omega^1(V \bmod R)$. For an ordered tree, we define ω_Γ as $\omega_\Gamma = \omega_{e_r} \wedge \cdots \wedge \omega_{e_1}$ in $\Omega(V \bmod R)$, where $E_\Gamma = \{e_1, \dots, e_r\}$ and $e_1 < \cdots < e_r$. It is easy to see the following lemma.

Lemma 3.1. *Assume $\#E = \#V - \#R$. Then Γ is a tree if and only if $\omega_\Gamma \neq 0$*

Let R be a sub set of $[n]$ such that $\{1, 2\} \subset R$. We define an ordering of $[n]$ by $1 \ll n \ll \cdots \ll 3 \ll 2$. We define $D(R)$ by $\{(x_1, \dots, x_i)_{i \in R} \mid x_i < x_j \text{ for } i \ll j\}$. For two subsets V and R of $[n]$ such that $R \subset V$, the fiber of the map $D(V) \rightarrow D(R)$ at $(x_i)_{i \in R} \in D(R)$ is denoted by $D(V/R, x_i)_{i \in R}$. Let $\alpha_{i,j}$ ($i, j \in V$) be positive real numbers. We choose a branch of $\Phi(V) = \prod_{i \ll j} (x_j - x_i)^{\alpha_{i,j}}$ on $D(V)$ with $\Phi \in \mathbf{R}_+$. For an ordered rooted graph Γ whose root set is R , we define a funcion $S_\Gamma = S_\Gamma(V/R, x_i)_{i \in R}$ on $D(R)$ as

$$S_\Gamma(V/R, x_i)_{i \in R} = \int_{D(V/R, x_i)_{i \in R}} \Phi(V) \prod_{(i,j) \in E_\Gamma} \alpha_{i,j} \omega_\Gamma$$

If R is fixed, it is denoted by S_Γ . Then S_Γ is a function on $(x_i)_{i \in R}$ and $\alpha_{i,j}$. The free abelian group generated by ordered rooted graphs whose root set and the set of vertices are R and V , is denoted by $\Gamma(V, R)$. For an element $\gamma = \sum a_\Gamma \Gamma$ in $\Gamma(V, R)$, we define S_γ by $S_\gamma = \sum a_\Gamma S_\Gamma$. The function S_γ is called the Selberg integral for γ .

Before the presentation of the main theorem, we introduce several combinatorial notions. For two natrual numbers n, r such that $2 \leq r \leq n$, we set $R = [r]$ and $V = [n]$. For an ordered rooted graph Γ , whose vertex set and root set are $[n]$ and $[r]$, we define an element $\Gamma \wedge (n+1, i)$ in $\Gamma([n+1], [r])$ for $i \in [n]$ according to the following recipe.

1. Choose a subset A of edges which are connecting to i . (A may be an emptyset.)
2. Replace the number i by $n+1$ for all the edges contained in A choosen in 1.
3. Make a graph Γ_A by adding the edge $(n+1, i)$ to the graph Γ and extend the original ordering to that of the edge set of Γ_A such that $(n+1, i)$ is the biggest edge.
4. Consider the sum $\sum_A \Gamma_A$ of Γ_A , where A runs through all the subsets of edges connecting to i . This summation is denoted by $\Gamma \wedge (n+1, i)$.

We extend the operation $\wedge(n+1, i_{n+1})$ from $\Gamma([n], [r])$ to $\Gamma([n+1], [r])$ by linearity. For an element $\gamma \in \Gamma([l], [r])$ and $(l+1, i_{l+1}), \dots, (n, i_n)$, where

$i_{l+1} \in [l], \dots, i_n \in [n-1]$, we define $\gamma \wedge (l+1, i_{l+1}) \wedge \dots \wedge (n, i_n)$ inductively by

$$\gamma \wedge (l+1, i_{l+1}) \wedge \dots \wedge (n, i_n) = (\gamma \wedge (l+1, i_{l+1}) \wedge \dots \wedge (n-1, i_{n-1})) \wedge (n, i_n)$$

The graph Γ with $V_\Gamma = R$ and $E_\Gamma = \emptyset$ is denoted by $\emptyset(R)$. A graph is denoted by $e_1 e_2 \dots e_b$, where the set of edges is $\{e_1 < e_2 < \dots < e_b\}$.

Example 3.2. If $R = \{1, 2\}$, $i_3 = 2$, $i_4 = 2$, then

$$\emptyset(R) \wedge (3, 2) \wedge (4, 2) = (2, 3) \wedge (4, 2) = (2, 3)(4, 2) + (4, 3)(4, 2)$$

We state the main theorem. Let H_α be a homogeneous MZV ring for $\mathbf{Q}\langle\langle\alpha_{i,j}, \alpha_{1,k}, \alpha_{2,k}\rangle\rangle_{3 \leq i,j,k \leq n, i \neq j}$.

Theorem 3.3. Let $R = \{1, 2\}$. For any $i_3 \in [2], \dots, i_n \in [n-1]$, put $\gamma = \emptyset(R) \wedge (3, i_3) \wedge \dots \wedge (n, i_n)$. Then $S_\gamma([n]/[2], 0, 1)$ is a holomorphic function on $\alpha_{i,j}$ and an element of H_α .

3.2. Differential equation satisfied by Selberg integral. First we compute the higher direct image of the connection on the configuration space $X_n = \{(x_1, \dots, x_n) \mid x_i \neq x_j \text{ for } i \neq j\}$ for distinct n -points in \mathbf{C} for the morphism $\pi : X_n \rightarrow X_{n-1}$ defined by $(x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{n-1})$. Let $A_{i,j} \in M(d, \mathbf{C})$ be matrices for $1 \leq i \neq j \leq n$ satisfying the following relations. These relations are called pure braid relations.

1. $A_{i,j} = A_{j,i}$.
2. $[A_{i,j}, A_{k,l}] = 0$ for all distinct i, j, k, l .
3. $[A_{i,j} + A_{j,k}, A_{i,k}] = 0$ for all distinct i, j, k .

Then the matrix valued 1-form

$$\omega = \sum_{1 \leq i < j \leq n} A_{i,j} d \log(x_i - x_j)$$

defines an integrable connection ∇ on $\mathcal{O}_{X_n}^d = \{v = {}^t(v_1, \dots, v_d)\}$ by

$$\nabla v = dv - \omega v.$$

Let v be a horizontal section of the connection ∇ on $D([n])$, i.e. $dv = \omega v$. For $i \in [n-1]$, and $(x_1, \dots, x_{n-1}) \in D([n-1])$, we define w_i as

$$w_i = \int_{D([n]/[n-1], x_1, \dots, x_{n-1})} \frac{A_{n,i}}{x_n - x_i} v dx_n.$$

Then w_i is a function on $(x_1, \dots, x_{n-1}) \in D([n-1])$. We have the following proposition.

Proposition 3.4. 1. $w_1 + \dots + w_{n-1} = 0$.

2. Let $W = {}^t(w_1, \dots, w_{n-1})$. Then W satisfies the differential equation

$$dW = \sum_{1 \leq i < j \leq n-1} \frac{A'_{i,j}(dx_i - dx_j)}{x_i - x_j} W,$$

where

$$(3.1) \quad A'_{i,j} = \begin{pmatrix} A_{ij} & \dots & \dots & 0 \\ \vdots & A_{ij} + A_{nj} & -A_{ni} & \vdots \\ \vdots & -A_{nj} & A_{ij} + A_{ni} & \vdots \\ 0 & \dots & \dots & A_{ij} \end{pmatrix} \begin{matrix} i \\ j \end{matrix}$$

Proof. 1. By the equality

$$\frac{\partial v}{\partial x_n} = \sum_{j=1}^{n-1} \frac{A_{nj}}{x_n - x_j} v,$$

and Stokes' theorem, the equality follows.

2. By using the differential equation for v , we have

$$\begin{aligned} & \frac{\partial}{\partial x_i} \left(\frac{A_{nj}}{x_n - x_j} v \right) \\ &= \frac{A_{nj}}{x_n - x_j} \left(\sum_{k \neq i, j, n} \frac{A_{ik} v}{x_i - x_k} + \frac{A_{ij} v}{x_i - x_j} + \frac{A_{ni} v}{x_i - x_n} \right) \\ &= \sum_{k \neq i, j, n} \frac{A_{ik} A_{nj} v}{(x_i - x_k)(x_n - x_j)} + \frac{A_{nj} v}{x_i - x_j} \left\{ -\frac{A_{ni} v}{x_n - x_i} + \frac{(A_{ij} + A_{ni}) v}{x_n - x_j} \right\}. \end{aligned}$$

By the commutativity condition of A_{ij} , we have

$$\frac{\partial}{\partial x_i} w_j = \sum_{k \neq i, j, 1 \leq k \leq n-1} \frac{A_{ik}}{x_i - x_k} w_k + \frac{1}{x_i - x_j} \{ (A_{ij} + A_{ni}) w_j - A_{nj} w_i \}$$

for $i \neq j$. Using the relation in 1., $\frac{\partial}{\partial x_i} w_i = \sum_{j \neq i, 1 \leq j \leq n-1} \frac{\partial}{\partial x_i} w_j$, and we obtain the statement of 2. \square

Remark 3.5. If A_{ij} satisfies the infinitesimal pure braid relations, then matrices A'_{ij} defined by (3.1) also satisfies the infinitesimal pure braid relations for $n-1$. Therefore the connection ∇' given by

$$\nabla' W = dw - \sum_{1 \leq i < j \leq n-1} \frac{(dx_i - dx_j) A'_{ij}}{x_i - x_j} W$$

on $(\mathcal{O}^{\oplus d})^{\oplus(n-1)}$ is integrable.

Let V_n be the local system of horizontal sections of the connection ∇ and $V_n|_{\pi^{-1}(x_1^0, \dots, x_{n-1}^0)}$ its restriction of the fiber $\pi^{-1}(x_1^0, \dots, x_{n-1}^0)$ of $\pi : X_n \rightarrow X_{n-1}$. Then the Euler-Poincare characteristic of $V_n|_{\pi^{-1}(x_1^0, \dots, x_{n-1}^0)}$ is $-(\text{rank } V) \cdot (n-2)$. Therefore under certain non-resonance condition, $\dim H^1(\pi^{-1}(x_1^0, \dots, x_{n-1}^0), V_n)$ is equal to $\text{rank } V \cdot (n-2)$. By a direct computation, the submodule $(\mathcal{M})^{\text{red}} = \{W = {}^t(w_1, \dots, w_{n-1}) \mid w_i \in \mathcal{O}^{\oplus d}, \sum_{i=1}^{n-1} w_i = 0\}$ comes to be a sub connection of $\mathcal{M} = ((\mathcal{O}^{\oplus d})^{\oplus(n-1)}, \nabla')$. As a consequence, horizontal section of $(\mathcal{M})^{\text{red}}$ is equal to the higher direct image of V

under the projection π . This construction is compatible with the sub local system in V .

We apply this inductive formula to compute the differential equation satisfied by Selberg integral. Note that the similar computation is executed in [1] with a different base of de Rham cohomology. Selberg integral is holomorphic with respect to α_{ij} for our base. This base is nothing but the β -nbc base introduced in Falk-Terao [3].

Let R be a ring. For a set of elements $\mathbf{a} = \{a_{pq}\}_{1 \leq p < q \leq k}$ satisfying the infinitesimal pure braid relation, we define a set of elements $\text{Ind}(\mathbf{a}) = \{\text{Ind}(\mathbf{a})_{ij}\}_{1 \leq i < j \leq k-1}$ in $M(k-1, R)$ by

$$\text{Ind}(\mathbf{a})_{ij} = \begin{pmatrix} a_{ij} & \dots & \dots & 0 \\ \vdots & a_{ij} + a_{kj} & -a_{ki} & \vdots \\ \vdots & -a_{kj} & a_{ij} + a_{ki} & \vdots \\ 0 & \dots & \dots & a_{ij} \end{pmatrix} \begin{matrix} i \\ j \end{matrix}.$$

Let $2 \leq r \leq n$ be integers and $V_{r,n}$ be a \mathbf{C} vector space of dimension $r(r+1) \cdots (n-1)$ whose coordinates are given by v_{i_{r+1}, \dots, i_n} for $1 \leq i_{r+1} \leq r, \dots, 1 \leq i_n \leq n-1$. We define $\mathbf{A}^{(p)} = \{A_{ij}^{(p)}\}_{1 \leq i < j \leq p}$ for $p = r, \dots, n-1$ by

$$\mathbf{A}^{(p)} = \text{Ind}(\mathbf{A}^{(p+1)})$$

and $A_{ij}^{(n)} = a_{ij}$. We define $V_{k,n}$ valued function $S^{(k)}(x_1, \dots, x_k)$ on $D([k])$ inductively by

$$S^{(k)} = \begin{pmatrix} \int_{D([k+1]/[k], x_i)_{i \in [k]}} \frac{A_{k+1,1}^{(k+1)}}{x_{k+1} - x_1} S^{(k+1)}(x_1, \dots, x_{k+1}) dx_{k+1} \\ \vdots \\ \int_{D([k+1]/[k], x_i)_{i \in [k]}} \frac{A_{k+1,k}^{(k+1)}}{x_{k+1} - x_k} S^{(k+1)}(x_1, \dots, x_{k+1}) dx_{k+1} \end{pmatrix}$$

for $k = r, \dots, n-1$ and

$$S^{(n)} = \prod_{1 \leq i < j \leq n} (x_j - x_i)^{\alpha_{ij}}.$$

We have the following corollary of Proposition 3.4.

Corollary 3.6. *The $V_{k,n}$ valued function $S^{(k)}$ satisfies the following differential equation*

$$dS^{(k)} = \Omega_k S^{(k)},$$

where $\Omega_k = \sum_{1 \leq i < j \leq k} \frac{A_{ij}^{(k)} d(x_i - x_j)}{x_i - x_j}$.

The next proposition is used at the proof of Main theorem 3.3.

Proposition 3.7. *Let $S_{i_{r+1}, \dots, i_n}^{(r)}$ be the (i_{r+1}, \dots, i_n) -component of $S^{(r)}$. Then we have*

$$(3.2) \quad \sum_{i'_p=1}^{p-1} S_{i_{r+1}, \dots, i_{p-1}, i'_p, i_{p+1}, \dots, i_n} = 0.$$

Proof. If $p = r+1$, then it is nothing but the first statement of Proposition 3.4. Suppose $p > r+1$. Then $(i_{r+1}, \dots, i_{p-1})$ -part of $S^{(r)}$ is a linear combination of

$$(3.3) \quad \int \prod_{i=r+1}^{p-1} A_{p_i q_i}^{(p)} \prod_{j=r+1}^{p-1} \frac{1}{x_j - x_{i_j}} S^{(p)} dx_{r+1} \cdots dx_{p-1}$$

Since the set $\{(a_{i_p, \dots, i_n}) \mid \sum_{i'_p=1}^{p-1} a_{i'_p, \dots, i_n} = 0\}$ is stable under the action of $A_{ab}^{(p)}$, (3.3) satisfies the relation $\sum_{i'_p=1}^{p-1} a_{i'_p, \dots, i_n} = 0$. \square

4. COMBINATORIAL PRELIMINARIES

4.1. Statement of the main theorem. In this section, we present combinatorial facts which is used to the computataion of Selberg integrals. Let P_n be the non-commutative ring $\mathbf{C}[a_{ij}]$ with the generators a_{ij} ($1 \leq i, j \leq n$) and the infinitesimal pure braind relations. We define a set of matrices $\mathbf{A}^{(k)} = \{A_{ij}^{(k)}\}_{1 \leq i, j \leq k}$ in $M(k(k+1) \cdots (n-1), P_n)$ inductively by the relations:

$$\mathbf{A}^{(k)} = \text{Ind}(\mathbf{A}^{(k+1)})$$

for $k = r, \dots, n-1$ and $A_{ij}^{(n)} = a_{ij}$. We introduce the degree of P_n by $\deg a_{ij} = 1$. Then the matrix elements of $A_{ij}^{(k)}$ are degree 1 for $k = r, \dots, n$ and $A_{ij}^{(k)}$ satisfies the pure braid relations. In other words, A ring homomorphism $P_r \rightarrow M(r(r+1) \cdots (n-1), P_n)$ is defined by attaching $A_{ij}^{(r)}$ to $a_{ij} \in P_r$. We define a vector $w_k \in P_n^{k(k+1) \cdots (n-1)} \otimes \mathbf{C}[\frac{1}{x_i - x_j}]$ inductively by the relation

$$(4.1) \quad w_k = \begin{pmatrix} \frac{A_{k+1,1}^{(k+1)}}{x_{k+1} - x_1} w_{k+1} \\ \vdots \\ \frac{A_{k+1,k}^{(k+1)}}{x_{k+1} - x_k} w_{k+1} \end{pmatrix}$$

for $k = r, \dots, n-2$ and

$$w_{n-1} = \begin{pmatrix} \frac{A_{n,1}^{(n)}}{x_n - x_1} \\ \vdots \\ \frac{A_{n,n-1}^{(n)}}{x_n - x_{n-1}} \end{pmatrix}.$$

In this section, we express each coordinate of w_r in terms of combinatorics introduced in §3.1. For an ordered tree Γ with the vertex set $[k]$ and root set $R = [r]$, we define $A_\Gamma^{(k)}$ by

$$A_\Gamma^{(k)} = \prod_{i=l}^1 A_{p_i, q_i}^{(k)} \in M(k(k+1) \cdots (n-1), P_n),$$

where $E_\Gamma = \{e_1 < \cdots < e_l\}$ and $e_i = (p_i, q_i)$. Here we use the notation $\prod_{i=l}^1 a_i = a_l a_{l-1} \cdots a_1$ in a non-comutative ring. We define a matrix valued

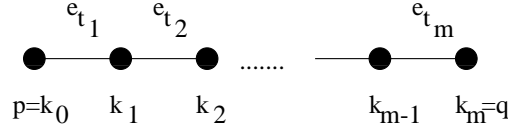


FIGURE 1.

differential form $\eta_\Gamma \in \Omega([k] \bmod [r]) \otimes M(k(k+1) \cdots (n-1), P_n)$ by $\eta_\Gamma = A_\Gamma^{(k)} \omega_\Gamma$, where ω_Γ is defined in §3.1. For an element $\gamma \in \Gamma([k], [r])$, we define η_γ by $\eta_\gamma = \sum_\Gamma a_\Gamma \eta_\Gamma$, where $\gamma = \sum_\Gamma a_\Gamma \Gamma$.

Theorem 4.1. *Let us denote the (i_{r+1}, \dots, i_n) -th coordinate of w_r by $w_r(i_{r+1}, \dots, i_n)$. Then*

$$w_r(i_{r+1}, \dots, i_n) dx_n \wedge \cdots \wedge dx_{r+1} = \eta_\gamma,$$

where $\gamma = \emptyset([r]) \wedge (r+1, i_{r+1}) \wedge \cdots \wedge (n, i_n)$.

The rest of this section is spent to prove Theorem 4.1.

4.2. Several lemmata. Let Γ be an ordered graph with the root set $[r]$ and the vertex set $[n-1]$. The edge set is denoted by $E = \{e_1 < \cdots < e_l\}$ and $e_i = (p_i, q_i)$. Suppose that p and q are contained in the same connected component. Then there exists unique path P connecting p and q in Γ . We write $P = \{e_{t_1}, \dots, e_{t_m}\}$. The subgraph P looks like figure 1.

Lemma 4.2. *Let $A_\Gamma^{(n-1)} \in M(n-1, P_n)$ be defined as in §4.1*

1. *If q -th component of*

$$(4.2) \quad A_\Gamma^{(n-1)} \begin{pmatrix} 0 \\ \vdots \\ a_{np} \\ \vdots \\ 0 \end{pmatrix} \in P_n^{(n-1)}$$

is not zero, then p and q are contained in the same connected component and $t_1 < t_2 < \cdots < t_m$.

2. *Suppose that $t_1 < t_2 < \cdots < t_m$. We write vertices of the path P as $p = k_0, k_1, \dots, k_m = q$, (see figure 1) and define B_i ($i = 1, \dots, l$) by*

$$B_i = \begin{cases} -a_{k_j, n} & (\text{if } i = t_j) \\ a_{p_i q_i} + a_{n q_i} & (\text{if } t_j < i < t_{j+1} \text{ and } e_i \text{ adjacent to } k_j \text{ and put } p_i = k_j) \\ a_{p_i q_i} & (\text{if } t_j < i < t_{j+1} \text{ and } e_i \text{ does not adjacent to } k_j) \end{cases}$$

(For the second case see figure 2.) Then q -th component of (1) is equal to $\prod_{i=1}^l B_i$.

Proof. For a vector $v = {}^t(v_1, \dots, v_{n-1}) \in P_n^{n-1}$, we set $\text{Supp}(v) = \{i \mid v_i \neq 0\}$.

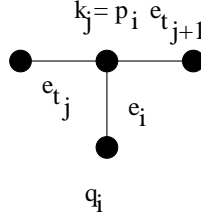


FIGURE 2.

1. If $\{i, j\} \cap \text{Supp}(v) = \emptyset$, then $\text{Supp}(A_{ij}^{(n-1)}v) = \text{Supp}(v)$ and k -th component of $A_{ij}^{(n-1)}v$ is equal to $a_{ij}v_k$ for $k \in \text{Supp}(v)$.
2. If $\{i, j\} \cap \text{Supp}(v) = \{i\}$, then $\text{Supp}(A_{ij}^{(n-1)}v) \subset \text{Supp}(v) \cup \{j\}$ and j -th component and i -th component of $A_{ij}^{(n-1)}v$ is equal to $-a_{nj}v_i$ and $(a_{ij} + a_{nj})v_i$ respectively.

Therefore if we define S_i inductively by

$$S_{i+1} = \begin{cases} S_i & (\text{if } e_{i+1} \cap S_i = \emptyset) \\ S_i \cup \{j\} & (\text{if } e_{i+1} \cap S_i = \{k\} \text{ and } e_{i+1} = \{j, k\}), \end{cases}$$

and $S_0 = \{p\}$. Then $\text{Supp}(\prod_{i=l}^1 A_{p_i, q_i}^{(n-1)})v \subset S_l$. If $q \in S_l$, then $t_1 < \dots < t_m$. This proves (1). For (2), we can prove

$$\left[\left(\prod_{i=s}^1 A_{p_i, q_i}^{(n-1)} \right) \begin{pmatrix} 0 \\ \vdots \\ a_{nk} \\ \vdots \\ 0 \end{pmatrix} \right]_{k_j} = a_{n, k_j} \cdot \prod_{j=s}^1 B_j.$$

if $t_j \leq s < t_{j+1}$ by induction on s using the infinitesimal pure braind relation (1) and (2). (In case $s \geq t_m$ and $s < t_1$, $a_{n, k_j} = a_{n, k_m}$ and $a_{n, k_j} = a_{n, k_0}$ respectively.) This complete 2. \square

Next we introduce an expression of $\emptyset([r]) \wedge (r+1, i_{r+1}) \wedge \dots \wedge (n, i_n)$ by using the notion of principal graph.

Definition 4.3. For an index set $I = (i_{k+1}, \dots, i_n)$, ($1 \leq i_p \leq p-1$), we define the ordered rooted graph P_I as follows.

1. The set of vertices is $\{1, \dots, n\}$,
2. the set of roots is $\{1, \dots, k\}$, and
3. the set of ordered edges is $\{(k+1, i_{k+1}) < \dots < (n, i_n)\}$.

The graph P_I is called the principal graph of I .

Let p, q be two vertices contained in the same connected component of P_I . The unique shortest path connecting p, q in P_I is denoted by $\gamma(p, q)$ and the

minimal edge of $\gamma(p, q)$ is denoted by $\min(p, q)$. Then by the construction of the principal graph, we have the following lemma.

Lemma 4.4. *Let us write a path $\gamma(p, q)$ connecting p, q in P_I as in figure 1: Suppose that e_{t_s} is the minimal edge of $\gamma(p, q)$. Then $t_1 > \dots > t_s < \dots < t_m$*

A graph Γ is called a support of $\gamma = \sum_{\Gamma} a_{\Gamma} \Gamma$, if $a_{\Gamma} \neq 0$. The set of supports of γ is denoted by $\text{Supp}(\gamma)$. Let $p, q \in [n]$ be vertices contained in the same connected component in P_I . We set $\gamma = \emptyset(\{1, \dots, k\}) \wedge (k+1, i_{k+1}) \wedge \dots \wedge (n, i_n)$. By the construction of γ , if $\Gamma \in \text{Supp}(\gamma)$ and (p, q) appears in Γ , then (p, q) is the m -th edge of Γ , where $e_m = \min(p, q)$. Conversely, for any pairs $(p_{k+1}, q_{k+1}), \dots, (p_n, q_n)$ such that

1. p_i and q_i are contained in the same connected component of P_I , and
2. $\min(p_j, q_j)$ is the j -th edge (j, i_j) of P_I ,

$a_{\Gamma} = 1$ for $\Gamma = (p_{k+1}, q_{k+1}) \cdots (p_n, q_n)$. We use distributive notation as

$$\begin{aligned} & \{(p_{k+1}, q_{k+1}) + (p'_{k+1}, q'_{k+1})\}(p_{k+2}, q_{k+2}) \cdots (p_n, q_n) \\ &= (p_{k+1}, q_{k+1})(p_{k+2}, q_{k+2}) \cdots (p_n, q_n) + (p'_{k+1}, q'_{k+1})(p_{k+2}, q_{k+2}) \cdots (p_n, q_n). \end{aligned}$$

Here the right hand side has a meaning as a formal linear combination of ordered graphs. The following proposition is nothing but the restatement of the definition of \wedge .

Proposition 4.5. *Let $S_i = \sum_{1 \leq p < q \leq n, \min(p, q) = e_i} \text{in } P_I (p, q)$. Then*

$$\emptyset(\{1, \dots, r\}) \wedge (r+1, i_{r+1}) \wedge \dots \wedge (n, i_n) = S_{r+1} \cdot S_{r+2} \cdots S_n$$

We finish this subsection by computing $\text{Res}_{x_n \rightarrow x_k}(\omega_{\Gamma})$ for an ordered rooted graph $\Gamma = \{e_{r+1}, \dots, e_n\} \in \text{Supp}(\gamma)$. Until the end of this subsection we assume $\Gamma \in \text{Supp}(\gamma)$ and (n, k) is an edge of Γ . Put $R_- = \{k' \mid (n, k') \in \Gamma, \min(n, k') < \min(n, k)\}$ and $R_+ = \{k' \mid (n, k') \in \Gamma, \min(n, k') \geq \min(n, k)\}$. We make a numbering of $R_+ = \{k_1 = i_n, k_2, \dots, k_s = k\}$ such that $\min(n, k_1) > \dots > \min(n, k_s)$. Set $e_{t_i} = \min(n, k_i)$. For the figure of principal graph see figure 3. If $s \geq 2$, we put $P = P(\Gamma, k)$ the power set of $R_+ - \{k_1, k_s\}$. For an element $p \in P$, we define a graph $\Gamma(p) \in \Gamma([n-1], [r])$ as follows. For $i = 2, \dots, s$, put $m(p, i) = \min\{j \mid k_j \in p \cup \{k_1\}, j < i\}$. The t_i -th edge of $\Gamma(p)$ is equal to (k_i, k_m) , where $m = m(p, i)$. The j -th edge is the same as Γ if $j \neq t_i, n$ ($i = 2, \dots, s$). A Set of ordered graph $\{\Gamma(p) \mid p \in P(\Gamma, k)\}$ is denoted by $R(\Gamma, k)$ and called the residue graph of Γ with respect to k .

Proposition 4.6. *If $\# | R_+ | \geq 2$, then*

$$\text{Res}_{x_n \rightarrow x_k}(\omega_{\Gamma}) = \sum_{p \in P(\Gamma, k)} (-1)^{\#p+1} \omega_{\Gamma(p)}.$$

Here the residue $\text{Res}_{x_n \rightarrow x_k} \omega = \eta|_{x_n=x_k}$, where $\omega = d \log(x_n - x_k) \wedge \eta$.

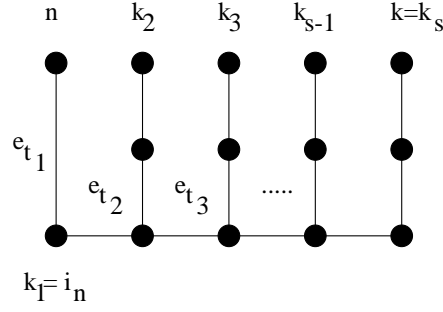


FIGURE 3.

Proof. We write $d \log(x_p - x_q) = \langle p, q \rangle$ for short. First we prove that

$$\begin{aligned} & \langle k, k_1 \rangle \cdots \langle k, k_{s-1} \rangle \\ &= \sum_{p \subset \{k_2, \dots, k_{s-1}\}} (-1)^{s+\#p} \langle m(p, 2), k_2 \rangle \cdots \langle m(p, s-1), k_{s-1} \rangle \langle m(p, s), k_s \rangle \end{aligned}$$

by induction on the cardinality of $\{k_1, \dots, k_{s-1}\}$. By the assumption of induction for $\{k_1, \dots, k_{s-2}\}$, we have

$$\begin{aligned} & \langle k, k_1 \rangle \cdots \langle k, k_{s-2} \rangle \langle k, k_{s-1} \rangle \\ &= \sum_{q \subset \{k_2, \dots, k_{s-1}\}} (-1)^{s+\#q} \langle m(q, 2), k_2 \rangle \cdots \langle m(q, s-1), k \rangle \langle k, k_{s-1} \rangle \\ &= \sum_{q \subset \{k_2, \dots, k_{s-1}\}} (-1)^{s+\#q} \langle m(q, 2), k_2 \rangle \cdots \langle m(q, s-2), k_{s-2} \rangle \\ & \quad (\langle m(q, s-1), k_{s-1} \rangle \langle k, k_{s-1} \rangle - \langle k_{s-1}, m(q, s-1) \rangle \langle k, m(q, s-1) \rangle) \end{aligned}$$

and the last expression gives the expression of $\{k_1, \dots, k_{s-1}\}$. Therefore we have

$$\begin{aligned} & \text{Res}_{x_n \rightarrow x_k} \langle n, k_1 \rangle \cdots \langle n, k_s \rangle \\ &= (-1)^{s-1} \langle n, k_1 \rangle \cdots \langle n, k_{s-1} \rangle \\ &= \sum_{p \subset \{k_2, \dots, k_{s-1}\}} (-1)^{\#p+1} \langle m(p, 2), k_2 \rangle \cdots \langle m(p, s-1), k_{s-1} \rangle \langle m(p, s), k_s \rangle. \end{aligned}$$

This implies the proposition. \square

4.3. Proof of Theorem 4.1. We prove Theorem 4.1 by induction. By Remark 3.5, the morphism

$$\rho : A_{ij}^{(n-1)} \mapsto \text{Ind}(\mathbf{a})_{ij}$$

defines a ring homomorphism from P_{n-1} to $M(n-1, P_n)$. We assume Theorem 4.1 for $n-1$. We define $W_k \in P_{n-1}^{r(r+1)\cdots(n-1)}$ for $k = 1, \dots, n-3$ inductively by the relation similar to (4.1) and

$$W_{n-2} = \begin{pmatrix} \frac{A_{n-1,1}^{(n-1)}}{x_{n-1}-x_1} \\ \vdots \\ \frac{A_{n-1,n-2}^{(n-1)}}{x_{n-1}-x_{n-2}} \end{pmatrix}.$$

Then by the assumption of induction,

$$\eta_\gamma = W_r(i_{r+1}, \dots, i_{n-1}) dx_{n-1} \wedge \cdots \wedge dx_{r+1}$$

for $\gamma = \emptyset(\{1, \dots, r\}) \wedge (r+1, i_{r+1}) \wedge \cdots \wedge (n-1, i_{n-1})$ in $P_{n-1} \otimes \Omega_R$. Here $W_r(i_{r+1}, \dots, i_{n-1})$ is the (i_{r+1}, \dots, i_n) -th component of W_r . By applying the above ring homomorphism ρ , we have

$$\rho(\eta_\gamma) = \rho(W_r(i_{r+1}, \dots, i_{n-1})) dx_{n-1} \wedge \cdots \wedge dx_{r+1}$$

in $M(n-1, P_{n-1}) \otimes \Omega_R$. By the definition of w_r , $w_r(i_{r+1}, \dots, i_n)$ is equal to the i_n -th component of the vector

$$\rho(W_r(i_{r+1}, \dots, i_{n-1})) \begin{pmatrix} \frac{a_{n1}}{x_n - x_1} \\ \vdots \\ \frac{a_{nn-1}}{x_n - x_{n-1}} \end{pmatrix}.$$

Therefore by taking the residue, $\text{Res}_{x_n \rightarrow x_k}$, it is enough to prove that

$$\rho(W_r(i_{r+1}, \dots, i_{n-1})) \begin{pmatrix} 0 \\ \vdots \\ a_{nk} \\ \vdots \\ 0 \end{pmatrix}_{i_n} = \text{Res}_{x_n \rightarrow x_k}(\eta_{\bar{\gamma}})$$

for all $k = 1, \dots, n-1$, where $\bar{\gamma} = \emptyset(\{1, \dots, k\}) \wedge (k+1, i_{k+1}) \wedge \cdots \wedge (n, i_n)$. We compute the left hand side and right hand side by using Lemma 4.2 and Proposition 4.6. Left hand side of (1) is expressed as a linear combination of η_Γ , where Γ is a support of γ . On the other hand, the expression given in Proposition 4.6 gives an expression of the right hand side by a linear combination of η_Γ , where Γ is a support of γ . By comparing the coefficient of ω_Γ it is enough to prove the following proposition.

Proposition 4.7. *Let $\Gamma \subset \text{Supp}(\gamma)$ and k, i_n be contained in the same connected component of Γ .*

1. $\text{length}(k, i_n) = \#p + 1$ if $\bar{\Gamma}(p) = \Gamma$. Here $\text{length}(k, i_n)$ is the length of the path connecting k and i_n in Γ .
- 2.

$$(-1)^{\text{length}(k, i_n)} \prod_{i=r+1}^n B_i = \sum_{\{\bar{\Gamma} \in \text{Supp}(\bar{\gamma}) | \Gamma \in R(\bar{\Gamma}, k)\}} A_{\bar{\Gamma}}^{(n)},$$

were B_i is defined in Lemma 4.2 and $R(\bar{\Gamma}, k)$ is defined in Proposition 4.6.

Proof. Let $\Gamma \in \text{Supp } \gamma$ and suppose $R(\bar{\Gamma}, k) \ni \Gamma$. As in Lemma 4.2, we make the numbering of the path in Γ from k to i_n as $e_{t_1} = (n, k_1), e_{t_2} = (k_1, k_2), \dots, e_{t_m} = (k_{s-1}, k)$ and $k_s = k, k_1 = i_n$. First we claim the set $L = L(\bar{\Gamma}) = \{l \mid (n, l) \in \bar{\Gamma}\}$ contains k_1, \dots, k_s . Since $\text{Res}_{x_n \rightarrow x_k}(\bar{\Gamma})$ is not zero, $\bar{\Gamma}$ contains (n, k) , i.e. $k = k_s \in L$. If the path connecting k and i_n in the corresponding graph $\bar{\Gamma}(p)$ is k_1, \dots, k_s , then $p = \{k_2, \dots, k_{s-1}\}$. Therefore $L \supset \{k_1, \dots, k_s\}$. If $l \in L - \{k_1, \dots, k_s\}$ and q is a minimal element satisfying $\min(i_n, l) < \min(i_n, k_q)$, then $\bar{\Gamma}(p)$ contains an edge (l, k_q) by the definition of $\bar{\Gamma}(p)$, i.e. $(l, k_q) \in G(\Gamma, k, i_n)$, where

$$G(\Gamma, k, i_n) = \{e : \text{edge} \mid \text{There exists } i \text{ such that } e \ni k_i, \\ \min(k_{i-1}, i_n) \leq e < \min(k_i, i_n)\}.$$

Therefore $L - \{k_1, \dots, k_s\} \subset G(\Gamma, k, i_n)$.

Conversely, for any subset L of $\{1, \dots, n\}$ satisfying

1. L is contained in the same connected component of i_n ,
2. $L \supset \{k_1, \dots, k_s\}$, and
3. $L - \{k_1, \dots, k_s\} \subset G(\Gamma, k, i_n)$,

there exists unique $\bar{\Gamma}(L)$ satisfying

1. $L(\bar{\Gamma}(L)) = L$,
2. $\bar{\Gamma} \in \text{Supp}(\bar{\gamma})$, and
3. $\text{Supp}(\text{Res}_{x_n \rightarrow x_k}(\bar{\Gamma})) \ni \Gamma$.

Therefore

$$\sum_{\{\bar{\Gamma} \mid \Gamma \in R(\bar{\Gamma}, k), \bar{\Gamma} \in \text{Supp}(\bar{\gamma})\}} A_{\bar{\Gamma}}^{(n)} = \sum_{\substack{L \supset \{k_1, \dots, k_s\}, L - \{k_1, \dots, k_s\} \subset G(\Gamma, k, i_n), \\ L \text{ is contained in the same} \\ \text{connected component of } i_n}} A_{\bar{\Gamma}(L)}^{(n)} \\ = (-1)^{\text{length}(k, i_n)} \prod_{i=k+1}^n B_i$$

□

5. PROOF OF THE MAIN THEOREM

5.1. Some lemmata for the assypmtotic beheaviors. In this subsection, we investigate the asymptotic behavior of the solution of linear differential equation with regular singularity. Let $A \in \frac{1}{x}M(d, \mathcal{O}_x)$, where \mathcal{O}_x is a germ of holomorphic functions at $x = 0$. We are interested in the differential equation for $r \times r$ -matrix valued function V :

$$\frac{dV}{dx} = AV.$$

We write $A = Rx^{-1} + \sum_{i=0}^{\infty} A_i x^i$, where $R, A_i \in M(r, \mathbf{C})$. If all the eigen values of R are enough small, then the solution V can be written as $V = Fx^R C_0$, where F is an $r \times r$ -valued homolorphic function in $I + xM(r, \mathcal{O}_x)$,

and $C_0 \in GL(r, \mathbf{C})$. In the rest of this section, we assume that all the eigen values of R are sufficiently small positive real numbers and R is semi-simple. The eigen value of R is denoted by $0 < \lambda_1 < \dots < \lambda_s$.

Lemma 5.1. *Let $\mathbf{C}^r = \oplus_{i=1}^s W_i$ be the eigen space decomposition of \mathbf{C}^r with respect to R .*

1. *If $w_i \in W_i$, then all the element a_k of the vector $Fx^R w_i$ satisfies the estimation $|a_k| \leq |x|^{\lambda_i} c$ with some constant c for $k = 1, \dots, r$. Moreover we have $\lim_{x \rightarrow 0} (x^{-\lambda_i} Fx^R w_i) = w_i$.*
2. *Let $\lambda > \lambda_i$. If $w_i \in W_i$ and all the elements a_k of $Fx^R w_i$ satisfy $|a_k| \leq x^\lambda c$ with some constant c , then $w = 0$.*
3. *Let λ_i be an eigenvalue of R . Let $p : W_i \rightarrow \mathbf{C}^l$ be a linear map and the composite $\mathbf{C}^r \rightarrow W_i \rightarrow \mathbf{C}^l$ is denoted by \tilde{p} . Then we have*

$$\tilde{p}(\lim_{x \rightarrow 0} x^{-\lambda_i} Fx^R w) = \tilde{p}(\lim_{x \rightarrow 0} x^{-R} Fx^R w)$$

for any $w \in W$.

Proof. Since $F = I + xm$, $m \in M(r, \mathcal{O}_x)$, using identity $\lim_{x \rightarrow 0} x^{-\lambda} x m x^R = \lim_{x \rightarrow 0} x^{-R} x m x^R = 0$, we get the statements. \square

Let n, k be integers such that $2 \leq k \leq n$ and we define reduced part $V^{red} = V_k^{red}$ as in §3.2. The restriction of $A_{ij}^{(k)}$ to V^{red} is denoted by $A_{ij,red}^{(k)}$. For a subset S of $[i, k]$, we define $A_S^{(k)}$ and $A_{S,red}^{(k)}$ by

$$A_S^{(k)} = \sum_{i < j, i, j \in S} A_{ij}^{(k)}, A_{S,red}^{(k)} = \sum_{i < j, i, j \in S} A_{ij,red}^{(k)}.$$

From now on, a_{ij} is sufficiently generic small positive real number. For a semisimple matrix A , the formal sum of eigen values of A counting their multiplicities is denoted by $\sigma(A)$: $\sigma(A) = \sum(\text{eigen values of } A)$. In this situation, the set of eigen values is denoted by $\text{Supp}(\sigma(A))$.

Proposition 5.2. *Under the notations and assumptions as above, $A_S^{(k)}$ and $A_{S,red}^{(k)}$ are semi-simple and*

$$\begin{aligned} \sigma(A_S^{(k)}) &= \sum_{T \subset [k+1, n]} (k - l; |T^c|)(l; |T|) a_{S \cup T}, \\ \sigma(A_{S,red}^{(k)}) &= \sum_{T \subset [k+1, n]} (k - l - 1; |T^c|)(l; |T|) a_{S \cup T}, \end{aligned}$$

where $a_U = \sum_{i < j, i, j \in U} a_{ij}$ for a subset $U \subset [1, n]$. For a subset $T \subset [k+1, n]$, $T^c = [k+1, n] - T$ and $l = \# |S| - 1$ and $(a; b) = a(a+1) \cdots (a+b-1)$.

To prove the above proposition, we use the following two elementary lemmata.

Lemma 5.3. *Let X be $kN \times kN$ -matrix. We assume that there exist semi-simple matrices B and D and matrices C_1, \dots, C_k such that*

$$A \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} i+1 = \begin{pmatrix} 0 \\ \vdots \\ B \\ -B \\ \vdots \\ 0 \end{pmatrix}$$

$$A \begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix} = \begin{pmatrix} C_1 D \\ \vdots \\ C_k D \end{pmatrix},$$

with $\text{Supp}(\sigma(B)) \cap \text{Supp}(\sigma(D)) = \emptyset$. Then

1. $\sigma(A) = (k-1)\sigma(B) + \sigma(D)$.
2. $(k-1)N$ -dimensional subvector space $V^{\text{red}} = \{(v_1, \dots, v_k) \mid v_i \in \mathbf{C}^N, \sum v_i = 0\}$ is stable under the action of A . Let A^{red} be the restriction of A to V^{red} . Then $\sigma(A^{\text{red}}) = (k-1)\sigma(B)$.

Lemma 5.4. *Let $a_{ij} \in P_k$ and set $A_{ij} = \text{Ind}(\mathbf{a})_{ij}$ for $1 \leq i < j \leq k-1$, $A_{[1,k-1]} = \sum_{1 \leq i < j \leq k-1} A_{ij}$, $a_{[1,k-1]} = \sum_{1 \leq i < j \leq k-1} a_{ij}$ and $a_{[1,k]} = \sum_{1 \leq i < j \leq k} a_{ij}$. Then we have*

$$A_{[1,k-1]} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} i+1 = \begin{pmatrix} 0 \\ \vdots \\ a_{[1,k]} \\ -a_{[1,k]} \\ \vdots \\ 0 \end{pmatrix}$$

$$A_{[1,k-1]} \begin{pmatrix} a_{k1} \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} a_{k1} a_{[1,k-1]} \\ \vdots \\ a_{kk-1} a_{[1,k-1]} \end{pmatrix}.$$

Proof. The first equality follows from the expression

$$A_{[1,k-1]} = \begin{pmatrix} a_{[1,k-1]} + \sum_{j \neq 1} a_{k,j} & -a_{k1} & \cdots \\ -a_{k2} & a_{[1,k-1]} + \sum_{j \neq 2} a_{kj} & \cdots \\ -a_{k3} & -a_{k3} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

The second equality is obtained directly by the equality

$$A_{ij} \begin{pmatrix} a_{k1} \\ \vdots \\ a_{kk-1} \end{pmatrix} = \begin{pmatrix} a_{k1}a_{ij} \\ \vdots \\ a_{kk-1}a_{ij} \end{pmatrix}.$$

□

Proof. (of Proposition 5.2) We prove the proposition by induction. By two lemmata, we have

$$\begin{aligned} \sigma(A_S^{(k)}) &= (k-l)\sigma(A_S^{(k+1)}) + l\sigma(A_{S \cup \{k+1\}}^{(k+1)}), \\ \sigma(A_{S,red}^{(k)}) &= (k-l-1)\sigma(A_{S,red}^{(k+1)}) + l\sigma(A_{S \cup \{k+1\},red}^{(k+1)}), \end{aligned}$$

using homomorphism $P^{(k+1)} \rightarrow M((k+1)(k+2) \cdots (n-1), \mathbf{C})$ and assumption of independence of a_{ij} . □

5.2. Relation between Selberg integral and Drinfeld associator. In this section, we will compare vectors whose elements are given by Selberg integrals with the Drinfeld associator. Let $n \geq 3$ be an integer and we define $A_{ij}^{(k)}$ as in §3.2. We set $V = V_{3,n} = \mathbf{C}^{3 \cdot 4 \cdots (n-1)}$. Let $S = S([n]/[3], x_1, x_2, x_3, \alpha_{ij})$ be a V -valued function on x_1, x_2, x_3 whose (i_4, \dots, i_n) -component is given by $S_{\emptyset(\{1,2,3\}) \wedge (4,i_4) \cdots (n,i_n)}([n]/[3], x_1, x_2, x_3, \alpha_{ij})$. Then S satisfies the differential equation

$$dS = (A_{13}^{(3)} d \log(x_1 - x_3) + A_{23}^{(3)} d \log(x_2 - x_3))S.$$

We set $\bar{S}(x_3) = S([n]/[3], 0, 1, x_3)$. Then \bar{S} satisfies the equation

$$\frac{d\bar{S}}{dx_3} = (A_{13}^{(3)} \frac{dx_3}{x_3} + A_{23}^{(3)} \frac{dx_3}{x_3 - 1})\bar{S}.$$

Therefore by considering the rational representation ρ of degree 1 : $\rho : \mathbf{C}\langle\langle X, Y \rangle\rangle \rightarrow M(3 \cdot 4 \cdots (n-1), \mathbf{Q}[[\alpha_{ij}]])$, we have

$$\lim_{x \rightarrow 1} (1 - x_3)^{-A_{23}^{(3)}} \bar{S}(x_3) = \rho(\Phi(X, Y)) \lim_{x_3 \rightarrow 0} x_3^{-A_{13}^{(3)}} \bar{S}(x_3).$$

We have the following lemma.

Lemma 5.5. 1. For any i_4, \dots, i_n , we put $\gamma = \emptyset(\{1, 2, 3\}) \wedge (4, i_4) \wedge \cdots \wedge (n, i_n)$. Then for a sufficiently small x_3 , we have an estimation

$$(5.1) \quad |S_\gamma([n]/[3], 0, 1, x_3)| < cx_3^{\alpha_{max}}$$

for some constant c . Here α_{max} is the maximal eigenvalue $\sum_{1 \leq i < j \leq n, i, j \neq 2} \alpha_{ij}$ of $A_{13}^{(3)}$.

2. For $\Gamma \in \Gamma([n], [3])$,

$$\begin{aligned} & \lim_{x_3 \rightarrow 0} x_3^{-\alpha_{max}} S_\Gamma([n]/[3], 0, 1, x_3) \\ &= \begin{cases} S_{\Gamma'}([n] - \{2\}/\{1, 3\}, 0, 1) & \text{(if there is no edges containing 2)} \\ 0 & \text{(otherwise)} \end{cases} \end{aligned}$$

Here $\Gamma' \in \Gamma([n] - \{2\}, \{1, 3\})$ is the ordered graph obtained by deleting 2 from the graph Γ .

Proof. By Proposition 5.2, we have $\alpha_{max} = \sum_{1 \leq i < j \leq n, i, j \neq 2} \alpha_{ij}$. To prove the statement, it is enough to prove that

$$\int_D \prod_{1 \leq i < j \leq n} (x_i - x_j)^{\alpha_{ij}} \omega_\Gamma \big|_{x_1=0, x_2=1}$$

satisfies the estimation of (5.1) for an ordered rooted tree Γ with the root set $[3]$. We change variable by $x_p = \xi_p x_3$ for $p = 4, \dots, n$. Then

$$(5.2) \quad \omega_\gamma = \pm \prod_{(p_i, q_i) \in E_\Gamma, \text{ not adjacent to 2}} \frac{d\xi_{p_i} - d\xi_{q_i}}{\xi_{p_i} - \xi_{q_i}} \prod_{(p_i, 2) \in E_\Gamma} \frac{x_3 d\xi_{p_i}}{-1} \cdot (1 + o(1)).$$

and

$$\prod_{1 \leq i < j \leq n} (x_i - x_j)^{\alpha_{ij}} = \prod_{1 \leq i < j \leq n, i, j \neq 2} (\xi_i - \xi_j)^{\alpha_{ij}} \cdot x_3^{\alpha_{max}} (1 + o(1)).$$

Here we put $\xi_3 = 1, \xi_1 = 0$. In particular, $\lim_{x_3 \rightarrow 0} x_3^{-\alpha_{max}} \int_D \Phi \omega_\Gamma = 0$ if Γ contains an edge adjacent to 2. The signature in (5.2) arise from the substitution for separating edges of Γ adjacent to 2 and those does not adjacent to 2. If Γ contains no edges adjacent to 2, we get the second statement. \square

From Lemma 5.1, we have the following corollary.

Corollary 5.6. *The (i_4, \dots, i_n) -th component of $\lim_{x_3 \rightarrow 0} x_3^{-A_{13}^{(3)}} \bar{S}(x_3)$ is equal to $S_\gamma([n] - \{2\}/\{1, 3\}, 0, 1)$ if $i_p \neq 2$ for $p = 4, \dots, n$, where $\gamma = \emptyset(\{1, 3\}) \wedge (4, i_4) \wedge \dots \wedge (n, i_n)$ and 0 otherwise.*

Proof. By the definition of $\gamma = \emptyset([3]) \wedge (4, i_4) \wedge \dots \wedge (n, i_n)$, if $i_p = 2$ for some p , then any $\Gamma \in \text{Supp}(\gamma)$ has an edge adjacent to 2. If $i_p \neq 2$ for all p , then any $\Gamma \in \text{Supp}(\gamma)$ contains no edges adjacent to 2. Therefore the statement follows from Proposition 5.5. \square

Next we consider the assumptotic behavior for $x_3 \rightarrow 1$. Let I be the set $\{(i_4, \dots, i_n) \mid i_p \neq 2, 3\}$. By the definition of $A_{23}^{(3)}$, the projection $p : V \rightarrow \mathbf{C}^I$ to I -th coordinate factors through α_{23} eigen projection. Therefore we have

$$p(\lim_{x_3 \rightarrow 1} (1 - x_3)^{-A_{23}^{(3)}} \bar{S}(x_3)) = p(\lim_{x_3 \rightarrow 1} (1 - x_3)^{-\alpha_{23}} \bar{S}(x_3)).$$

by Lemma 5.1. On the other hand, it is easy to see the following lemma.

Lemma 5.7. *If Γ contains no edges containing 2 and 3, then*

$$\lim_{x_3 \rightarrow 1} (1 - x_3)^{-\alpha_{23}} S_\Gamma([n]/[3], 0, 1, x_3) = S_{\Gamma'}([n] - \{3\}/[2], 0, 1, \alpha'_{ij})$$

where Γ' is the ordered graph obtained by deleting 3 from the graph Γ , and $\alpha_{ij} = \alpha'_{ij}$ if $i, j \neq 2$ and $\alpha'_{2j} = \alpha_{2j} + \alpha_{3j}$.

Definition 5.8. *The vector $\lim_{x_3 \rightarrow 0} x_3^{-A_{13}^{(3)}} \bar{S}(x_3)$ and $\lim_{x_3 \rightarrow 1} (1 - x_3)^{-A_{23}^{(3)}} \bar{S}(x_3)$ is denoted by $V^{(1)}$ and $V^{(2)}$ respectively. Then we have*

$$(5.3) \quad p(V^{(2)}) = p(\rho(\Phi(X, Y))V^{(1)})$$

By Lemma 5.7, the (i_4, \dots, i_n) -th component of $V^{(2)}$ with $i_p \neq 2, 3$ is equal to $S_\gamma(0, 1, \alpha'_{ij})$, where $\alpha'_{ij} = \alpha_{ij}$ if $i, j \neq 2$ and $\alpha'_{2,j} = \alpha_{2j} + \alpha_{3j}$, where $\gamma = \emptyset(\{1, 2\}) \wedge (4, i_4) \wedge \dots \wedge (n, i_n)$. We compute the limit of all the component of $V^{(1)}$ for the limit $\alpha_{3i} \rightarrow 0$. For this purpose, we compute $\lim_{\alpha_{3i} \rightarrow 0} S_\gamma(\alpha_{ij})$ for $\gamma = \emptyset(\{1, 3\}) \wedge (4, i_4) \wedge \dots \wedge (n, i_n)$ with $i_p \neq 2$, in the next subsection.

5.3. Limit for $\alpha_{3i} \rightarrow 0$. In this subsection, we change numbering from that of the last subsection. Let Γ be an ordered graph with the root set $[2]$ and vertex set $[n]$. We set $\Phi = \prod_{i \ll j} (x_i - x_j)^{\alpha_{ij}}$, and

$$S(\alpha_{ij}) = \int_{D([n]/[2], 0, 1)} \eta_\Gamma \Phi.$$

Before proving Proposition 5.10, we remark the following lemma.

Lemma 5.9. *Let $F(x)$ be a continuous function defined on $(p, 1]$. Suppose that $F(x)$ is integrable on $(p, p + \epsilon]$. Then we have*

$$\lim_{\alpha \rightarrow 0} \int_p^1 \alpha(1 - x)^{\alpha-1} F(x) dx = F(1).$$

Proof. This is the fundamental property of δ -function $\lim_{\alpha \rightarrow 0} \alpha(1 - x)^{\alpha-1}$. \square

Proposition 5.10. 1. *If $\lim_{\alpha_{2i} \rightarrow 0} S(\alpha_{ij}) \neq 0$, then (1) Γ contains no edges adjacent to 2, or (2) $(2, 3)$ is a unique edge adjacent to 2.*
 2. *If $(2, 3)$ is a unique edge in Γ adjacent to 2, then $\lim_{\alpha_{2i} \rightarrow 0} S(\alpha_{ij})$ is equal to $S_{\Gamma'}(\alpha'_{ij})$, where Γ' is the ordered graph obtained by deleting the edge $(2, 3)$ and replace the numbering 3 of original edge by the new numbering 2 and $\alpha'_{ij} = \alpha_{ij}$ if $i, j \neq 2$ and $\alpha'_{2,k} = \alpha_{3,k}$.*

Proof. Suppose that Γ contains an edge adjacent to 2. Let $p \leq 3$ be the minimal number such that $(2, p)$ is an edge of Γ . Set

$$F(x_p, \dots, x_n) = \prod_{(pq) \in E_\Gamma, \neq (2,p)} a_{pq} \prod_{1 \leq i \leq n, p \leq j \leq n, i < j, (i,j) \neq (2,p)} (x_i - x_j)^{\alpha_{ij} + \epsilon_{ij}} \\ \int_{\{x_p < \dots < x_3 < 1\}} \prod_{1 \leq i < j \leq p-1} (x_i - x_j)^{\alpha_{ij} + \epsilon_{ij}} dx_{p-1} \dots dx_3,$$

where $\epsilon_{ij} = -1$ if (i, j) is an edge of Γ and 0 otherwise. Then

$$\lim_{\alpha_{2p} \rightarrow 0} S_\Gamma = \int_{\{0 < x_n < \dots < x_p < 1\}} F(x_p, \dots, x_n) \alpha_{2p} (1 - x_p)^{\alpha_{2p}-1}.$$

Therefore if $p \neq 3$ or there exist at least two p such that $(2, p)$ is an edge of Γ , then $S_\Gamma = 0$. If $p = 3$ and there is no edge adjacent to 2 other than $(2, 3)$, then $\lim_{\alpha_{23} \rightarrow 0} S_\Gamma = S_{\Gamma'}(\alpha'_{ij})$. \square

We define $S_\gamma(\alpha_{ij})$ by $\sum a_\Gamma S_\Gamma(\alpha_{ij})$, where $\gamma = \sum a_\Gamma \Gamma \in \Gamma([2], [n])$.

Corollary 5.11. *Let $\gamma = \emptyset(\{1, 2\}) \wedge (3, i_3) \wedge \dots \wedge (n, i_n)$.*

1. *If there exists $k \neq 3$ such that $i_k = 2$, then $\lim_{\alpha_{2i} \rightarrow 0} S_\gamma(\alpha_{ij}) = 0$*
2. *If $i_3 = 2$ and $i_k \neq 2$ for $k \neq 3$, then*

$$\lim_{\alpha_{2i} \rightarrow 0} S_\gamma(\alpha_{ij}) = S_{\gamma'}(\alpha'_{ij}),$$

where γ' is $\emptyset(\{1, 3\}) \wedge (4, i_4) \wedge \dots \wedge (n, i_n)$.

Proof. (Proof of the Main Theorem 3.3) We can proceed by the induction on n . We consider the limit of (5.3) for $\alpha_{3i} \rightarrow 0$. Then all the entries of $\lim_{\alpha_{3i} \rightarrow 0} (\rho(\Phi(X, Y)))$ are contained in H_α by Corollary 2.2. By Corollary 5.11, all the entries of $\lim_{\alpha_{3i} \rightarrow 0} V^{(1)}$ are contained in H_α . Therefore all the entries of $\lim_{\alpha_{3i} \rightarrow 0} p(V^{(0)})$ are also contained in H_α . Therefore $S_\gamma(0, 1, \alpha_{ij})$ is an element of H_α for $\gamma = \emptyset(\{1, 2\}) \wedge (3, i_3) \wedge \dots \wedge (n, i_n)$ under the restriction

$$(R) : i_k \neq 2 \text{ for all } k.$$

On the other hand, by the relation (3.2), the restriction (R) is not necessary. This completes the main theorem. \square

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